Laser: Theory and Modern Applications

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to homework No. 3

Attention: for some exercises the given solutions are not the same as found on the exercise sheet. The main calculation remains the same.

1 Fabry Perot Etalon

1. Finesse $\mathfrak{F} \triangleq \frac{\nu_{FSR}}{\delta\nu}$

The resonance frequencies of a Fabry-Perot resonator are where $\frac{\phi}{2} = m \cdot \pi$:

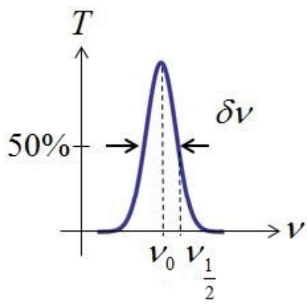
$$\phi = 2 \cdot k \cdot d = 2 \cdot \frac{\omega}{c} n \cdot d = \frac{4\pi\nu n d}{c}$$

$$\text{Resonance frequencies } \nu_m = \frac{mc}{2nd}, m \in \mathbb{N}$$

$$\text{Resonance frequencies } \omega_m = \frac{mc\pi}{nd}, m \in \mathbb{N}$$

$$\text{Free spectral range } FSR = \nu_{FSR} = \nu_{m+1} - \nu_m = \frac{c}{2nd}$$

Let's see just one resonance frequency ν_0



Full width at half maximum $\delta\nu = 2\nu_{\frac{1}{2}}$, where $\nu_{\frac{1}{2}}$ is the frequency for which $T = 50\%$

Transmission coefficient T :

$$T = \frac{I_T}{I_{in}} = \frac{1}{1 + F\sin^2(\phi/2)} = \frac{1}{2}$$

$$\Longrightarrow F\sin^2(\phi/2) = 1 \Longrightarrow \sin^2(\phi/2) = \frac{1}{F} \Longrightarrow \sin(\phi/2) = \sqrt{\frac{1}{F}}$$

$$F >> 1 \Longrightarrow \sin x \simeq x \Longrightarrow \phi/2 = \sqrt{\frac{1}{F}} = \frac{1-R}{2\sqrt{R}} \Longrightarrow \phi = \frac{1-R}{\sqrt{R}}$$

$$\Longrightarrow \frac{4\pi\nu_{\frac{1}{2}} nd}{c} = \frac{1-R}{\sqrt{R}} \Longrightarrow \delta\nu = 2\nu_{\frac{1}{2}} = \frac{1-R}{\sqrt{R}} \frac{c}{2\pi nd}$$

$$\text{Finesse } \mathfrak{F} \triangleq \frac{\nu_{FSR}}{\delta\nu} = \frac{c}{2nd} \cdot \frac{\sqrt{R}}{1-R} \frac{2\pi nd}{c} = \frac{\pi\sqrt{R}}{1-R}$$

2. Example:

$$n = 1.44 \implies R = 0.33$$

$$\nu_{FSR} = \frac{c}{2nd} = \frac{3 \times 10^8}{2 \cdot 1.44 \cdot 1 \times 10^{-3}} = 104.1 \text{ GHz}$$

3. If no mirror absorption: $T = 1 - R$. With absorption: $T = 1 - R - \alpha$

$$I_r = RI_{in},\ I_t = TI_{in},\ I_{abs} = \alpha I_{in},\ I_{in} = I_{abs} + I_r + I_t$$
$$E_r = \sqrt{R}E_{in},\ E_t = \sqrt{T}E_{in}$$

Assume two mirror are the same:

$$\begin{aligned}
T = \frac{I_t}{I_i} = \left|\frac{E_t}{E_i}\right|^2 = \frac{E_t E_t^*}{E_i E_i^*} &= \frac{(\sqrt{T_1 T_2}e^{-ikl})(\sqrt{T_1 T_2}e^{ikl})}{(1 - \sqrt{R_1 R_2}e^{-i2kl})(1 - \sqrt{R_1 R_2}e^{i2kl})} \\
&= \frac{T^2}{1 - Re^{i2kl} - Re^{-i2kl} + R^2} \longleftarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\
&= \frac{T^2}{1 - 2R\cos(2kl) + R^2} \longleftarrow 2\sin^2(\tfrac{\theta}{2}) = 1 - cos(\theta) \\
&= \frac{T^2}{1 - 2R(1 - 2\sin^2(kl)) + R^2} \\
&= \frac{T^2}{(1-R)^2 + 4R\sin^2(kl)} \longleftarrow F = \frac{4R}{(1-R)^2} \\
&= \frac{T^2/(1-R)^2}{1 + F\sin^2(kl)} \longleftarrow T = 1 - R - \alpha \\
&= \frac{(1-R-\alpha)^2}{(1-R)^2}\frac{1}{1 + F\sin^2(kl)}
\end{aligned}$$

4. In order to observe resonances with the peak transmission > 0.5:

$$\begin{aligned}
\frac{(1-R-\alpha)^2}{(1-R)^2} > 0.5 &\Rightarrow 2(1-\alpha-R)^2 > (1-R)^2 \\
&\Rightarrow 2(1-\alpha)^2 - 4(1-\alpha)R + 2R^2 > 1 - 2R + R^2 \\
&\Rightarrow R^2 + (-4(1-\alpha) + 2)R + 2(1-\alpha)^2 - 1 > 0 \\
&\Rightarrow R^2 + (4\alpha - 2)R + (1 - 4\alpha + 2\alpha^2) > 0 \\
R_{1,2} &= \frac{-(4\alpha - 2) \pm \sqrt{(4\alpha-2)^2 - 4(1 - 4\alpha + 2\alpha^2)}}{2} \\
&= \frac{-(4\alpha - 2) \pm 2\sqrt{2}\alpha}{2} = 1 - 2\alpha \pm \sqrt{2}\alpha
\end{aligned}$$

Then, the minimum reflectivity R necessary to have resonant transmission > 0.5, is

$$R \rightarrow 1 - 2\alpha - \sqrt{2}\alpha$$

Because, otherwise will produces a negative transmission which is not physical.

2 Input-output relations for an optical resonator

1. Input field is $E_{in}e^{-i\omega t}$, intracavity field is $Ee^{-i\omega t}$, spacing between the mirrors l

$$\begin{aligned} Ee^{-i\omega t} &= t_1 E_{in} e^{-i\omega t} + t_1 r_1 r_2 E_{in} e^{-i\omega t} e^{-i2kl} + t_1 r_1^2 r_2^2 E_{in} e^{-i\omega t} e^{-i4kl} + \cdots \\ &= t_1 E_{in} e^{-i\omega t} \cdot \left[1 + r_1 r_2 e^{-i2kl} + r_1^2 r_2^2 e^{-i4kl} + \cdots \right] \\ &= t_1 E_{in} e^{-i\omega t} \cdot \left[\frac{1}{1 - r_1 r_2 e^{-i2kl}} \right] \\ &= \frac{t_1}{1 - r_1 r_2 e^{-i\phi}} E_{in} e^{-i\omega t} \end{aligned}$$

$$\implies E = \frac{t_1}{1 - r_1 r_2 e^{-i\phi}} E_{in}$$

where

$$\phi = 2kl = 2\frac{\omega}{c} nl$$

2. We know that at resonance

$$\phi = m \cdot 2\pi = 2\frac{\omega_m}{c} nl \implies m = \frac{\omega_m nl}{\pi c}$$

Let's take a detuning ω_{det} close to cavity resonance frequencyω_m, $\Delta = \omega_{det} - \omega_m$

$$\phi = 2\frac{\omega_{det}}{c} nl = 2\frac{\omega_m + \Delta}{c} nl = 2\frac{\omega_m}{c} nl + 2\frac{\Delta}{c} nl = m \cdot 2\pi + \delta\phi$$

$$\implies \delta\phi = 2\frac{\omega_{det} - \omega_m}{c} nl$$

use 1:

$$e^{-i\phi} = e^{-i2\pi m} e^{-i\delta\phi} = e^{-i\delta\phi} \overset{(\delta\phi << 1)}{\simeq} 1 - i\delta\phi$$

use 2:

$$\delta\phi = \frac{2nl}{c}(\omega_{det} - \omega_m) = \frac{2nl}{c} \cdot \Delta$$

use 3:

$$R + T = 1 \longrightarrow r^2 + t^2 = 1 \longrightarrow r = \sqrt{1 - t^2} \longrightarrow r_1 r_2 = \sqrt{(1 - t_1^2)(1 - t_2^2)}$$

$$\frac{t_1}{r_1 r_2} = \sqrt{\frac{t_1^2}{1 - t_1^2 - t_2^2 - \cancel{t_1^2 t_2^2}}} \overset{(t << 1)}{\simeq} \sqrt{\frac{t_1^2}{1 - t_1^2 - t_2^2}} \simeq t_1$$

$$\frac{1 - r_1 r_2}{r_1 r_2} \simeq \frac{1 - \sqrt{1 - t_1^2 - t_2^2}}{\sqrt{1 - t_1^2 - t_2^2}} \simeq 1 - \sqrt{1 - t_1^2 - t_2^2} \overset{(\sqrt{1+x} = 1 + \frac{x}{2})}{\simeq} 1 - \left[1 - \frac{1}{2}(t_1^2 + t_2^2) \right] = \frac{1}{2}(t_1^2 + t_2^2)$$

Then we can derive:

$$\begin{aligned}
\frac{t_1}{1-r_1r_2e^{-i\phi}} &= \frac{t_1}{1-r_1r_2+ir_1r_2\delta\phi} \iff \\
&= \frac{t_1}{(1-r_1r_2)+ir_1r_2\frac{2nl}{c}\Delta} \\
&= \frac{\frac{t_1}{r_1r_2}\frac{c}{2nl}}{\frac{(1-r_1r_2)}{r_1r_2}\frac{c}{2nl}+i\Delta} \\
&= \frac{\frac{t_1}{r_1r_2}\frac{c}{2nl}}{\frac{(1-r_1r_2)}{r_1r_2}\frac{c}{2nl}+i\Delta}\cdot\frac{\sqrt{\frac{2nl}{c}}}{\sqrt{\frac{2nl}{c}}} \\
&= \frac{\frac{t_1}{r_1r_2}\sqrt{\frac{c}{2nl}}}{\frac{(1-r_1r_2)}{r_1r_2}\frac{c}{2nl}+i\Delta}\cdot\frac{1}{\sqrt{\frac{2nl}{c}}} \\
&\simeq \frac{\sqrt{t_1^2\frac{c}{2nl}}}{\frac{1}{2}(t_1^2+t_2^2)\frac{c}{2nl}+i\Delta}\cdot\frac{1}{\sqrt{\frac{2nl}{c}}} \\
&= \frac{\sqrt{\kappa_1}}{\frac{1}{2}(\kappa_1+\kappa_2)+i\Delta}\cdot\frac{1}{\sqrt{\frac{2nl}{c}}}
\end{aligned}$$

$$\implies \kappa_1 = \frac{ct_1^2}{2nl},\ \kappa_2 = \frac{ct_2^2}{2nl},\ \kappa = \kappa_1+\kappa_2 = \frac{c(t_1^2+t_2^2)}{2nl}$$

3. In class we saw photon life time τ_p

$$\tau_p = \frac{2nl}{c(1-R_1R_2)}$$

So the κ can be related to τ_p

$$\kappa = \frac{c(t_1^2+t_2^2)}{2nl} = \frac{c(1-R_1+1-R_2)}{2nl} \simeq \frac{c(1-R_1R_2)}{2nl} = \frac{1}{\tau_p}$$

Assumptions are as shown in the class:

$$\begin{aligned}
R_1 \approx 1,\ R_2 \approx 1 \implies & T_1\cdot T_2 \approx 0 \\
=&(1-R_1)(1-R_2)\approx 0 \\
=&1-R_1-R_2+R_1R_2 \approx 0 \\
\implies & 1-R_1-R_2 \approx -R_1R_2 \\
\implies & 2-R_1-R_2 \approx 1-R_1R_2
\end{aligned}$$

4.

$$E_{in}(t) = \int_{-\infty}^{\infty} E_{in}[\Omega]e^{-i\Omega t}d\Omega$$

$$a(t) = \sqrt{2dn/c}E(t)$$

$$\int_{-\infty}^{\infty}(i(\Delta-\Omega)+\kappa/2)a[\Omega]e^{-i\Omega t}d\Omega = \sqrt{\kappa_1}\int_{-\infty}^{\infty}E_{in}[\Omega]e^{-i\Omega t}d\Omega$$

$$\int_{-\infty}^{\infty} i\Omega a[\Omega]e^{-i\Omega t}d\Omega = -\frac{d}{dt}a(t)$$

$$(i\Delta + \kappa/2)a(t) + \frac{d}{dt}a(t) = \sqrt{\kappa_1}E_{in}(t)$$

$$\frac{d}{dt}a(t) = -(i\Delta + \kappa/2)a(t) + \sqrt{\kappa_1}E_{in}(t)$$

3 Multimode laser diodes : cavity length and Fabry-Perot analyzer

1. The mode spacing of a resonator of length l_{laser} is

$$\Delta\nu = \frac{c}{2nl_{laser}} \tag{1}$$

From the graph it can be seen

$$\Delta\lambda = \frac{0.4nm}{7} = 0.0057nm \tag{2}$$

In order to be able to use equation 1, we need to convert this wavelength bandwidth $\Delta\lambda$ into a frequency bandwidth $\Delta\nu$. It is incorrect to apply the relationship $\nu = \frac{c}{\lambda}$ to wavelength or frequency intervals (i.e. $\Delta\nu = \frac{c}{\Delta\lambda}$ does not hold). The correct formula for intervals can be found by differentiation:

$$\nu = \frac{c}{\lambda} \Rightarrow \delta\nu = -\frac{c}{\lambda^2}d\lambda$$

As long as the intervals remain small, the differentials can be replaced by finite differences ($d\lambda \implies \Delta\lambda$,$dv \implies \Delta\nu$). By dropping the minus sign (it indicates a change in the direction of the interval - this is not relevant here), and by substituting $\nu = \frac{c}{\lambda}$, we can obtain a formula that is easy to remember :

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda}$$

Starting from this formula, we can carry out the conversion as follows:

$$\frac{\Delta\nu}{\nu} = \frac{\Delta\lambda}{\lambda} \implies \Delta\nu = \frac{\Delta\lambda}{\lambda}\nu = \frac{\Delta\lambda}{\lambda^2}c \overset{\lambda=403nm}{\cong} 105GHz$$

From equation 1 :

$$l_{laser} = \frac{c}{2n\Delta\nu} = 550\mu m$$

2. Etalon Design The free spectral range of the Fabry-Perot is

$$\Delta\nu_{FSR} = \frac{c}{2nl_{etalon}}, \Delta\nu_{range} \geq 1nm$$

$$\Delta\nu_{FSR} = \Delta\nu_{range} = \frac{\Delta\lambda}{\lambda^2}c \geq \frac{(1\cdot10^{-9})(3\cdot10^{8})}{(405\cdot10^{-9})^2}$$

$$\implies l_{etalon} \leq \frac{c}{2n \cdot 1875GHz} \overset{n=1}{=} 82\mu m$$

The resolution of the Fabry-Perot etalon is taken as 10% of 105 GHz

$$\Delta\nu_{FWHMetalon} = 10.5GHz$$

This results in a required resonator finesse $\mathcal{F}$

$$\mathcal{F} = \frac{\Delta\nu_{FSR}}{\Delta\nu_{FWHM}} = \frac{1829GHz}{10.5GHz} = 178$$

The relationship between finesse $\mathcal{F}$ and mirror reflectivity can be described as

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R} \implies \mathcal{F}^2R^2 - R(2\mathcal{F}^2 + \pi^2) + \mathcal{F}^2 = 0$$

$$R \cong \frac{2\mathcal{F}^2 + \pi^2 \pm 2\mathcal{F}\pi}{2\mathcal{F}^2}$$

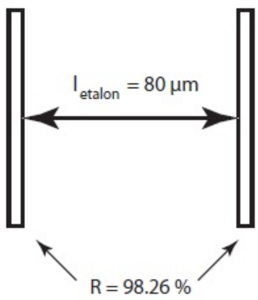
Since only (-) gives a physical value for R

$$R = 1 + \frac{\pi^2}{2\mathcal{F}^2} - \frac{\pi}{\mathcal{F}}$$

$$\implies R = 98.26\%$$

$$l_{etalon} = 82\mu m$$

$$n = 1(\text{in air})$$



3. Coherence length

$$l_{coh} = \frac{c}{n\Delta\nu} = \frac{(3 \cdot 10^8 m/s)}{(2.6)(736 \cdot 10^9 Hz)} = 156\mu m$$

4 Four Level System

1. Steady State

$$0 = -\Gamma_{10}\overline{N}_1 + \Gamma_{21}\overline{N}_2 \implies \overline{N}_1 = \frac{\Gamma_{21}}{\Gamma_{10}} \cdot \overline{N}_2$$

$$0 = P\overline{N}_0 - \Gamma_{21}\overline{N}_2 \implies \overline{N}_2 = \frac{P}{\Gamma_{21}} \cdot \overline{N}_0$$

$$0 = -P\overline{N}_0 + \Gamma_{10}\overline{N}_1 \implies \overline{N}_0 = \frac{\Gamma_{10}}{P} \cdot \overline{N}_1$$

from this follows

$$\overline{N}_2 = \frac{\Gamma_{10}}{\Gamma_{21}} \cdot \overline{N}_1 \tag{3}$$

$$\overline{N}_0 = \frac{\Gamma_{10}}{P} \cdot \overline{N}_1 \tag{4}$$

$$N_T = \overline{N}_0 + \overline{N}_1 + \overline{N}_2 \tag{5}$$

from equation (5) we get the following:

$$\frac{\Gamma_{10}}{P} \cdot \overline{N}_1 + \overline{N}_1 + \frac{\Gamma_{10}}{\Gamma_{21}} \cdot \overline{N}_1 = N_T$$

$$\overline{N}_1 \left(\frac{\Gamma_{10}}{P} + 1 + \frac{\Gamma_{10}}{\Gamma_{21}} \right) = N_T$$

$$\overline{N}_1 \left(\frac{\Gamma_{10}\Gamma_{21} + P\Gamma_{21} + P\Gamma_{10}}{P\Gamma_{21}} \right) = N_T$$

$$\overline{N}_1 = \frac{P\Gamma_{21}}{\Gamma_{10}\Gamma_{21} + P(\Gamma_{21} + \Gamma_{10})} N_T$$

$$(3) \implies \overline{N}_2 = \frac{P\Gamma_{21}}{\Gamma_{10}\Gamma_{21} + P(\Gamma_{21} + \Gamma_{10})} N_T$$

$$(4) \implies \overline{N}_0 = \frac{\Gamma_{10}\Gamma_{21}}{\Gamma_{10}\Gamma_{21} + P(\Gamma_{21} + \Gamma_{10})} N_T$$

subtracting N1 from N2 yields the solution:

$$\overline{N}_2 - \overline{N}_1 = \frac{P(\Gamma_{10} - \Gamma_{21})}{\Gamma_{10}\Gamma_{21} + P(\Gamma_{21} + \Gamma_{10})} N_T$$

2. Ratio between three and four level system Three level system:

$$\Delta N_T = \overline{N}_2 - \overline{N}_1 = \frac{P - \Gamma_{21}}{P + \Gamma_{21}} N_T$$

$$(P + \Gamma_{21})\Delta N_T = (P - \Gamma_{21}) N_T$$

$$P\Delta N_T + \Gamma_{21}\Delta N_T = P N_T - \Gamma_{21} N_T$$

$$\implies P(\Delta N_T - N_T) = -\Gamma_{21}(\Delta N_T + N_T)$$

$$(P)_{\text{3level threshold}} = \frac{\Gamma_{21}(\Delta N_T + N_T)}{N_T - \Delta N_T}$$

Four level system:

$$\Delta N_T = \frac{P(\Gamma_{10} - \Gamma_{21})}{\Gamma_{10}\Gamma_{21} + P(\Gamma_{21} + \Gamma_{10})} N_T$$

$$\Gamma_{10}\Gamma_{21}\Delta N_T + P\Delta N_T(\Gamma_{21} + \Gamma_{10}) = P(\Gamma_{10} - \Gamma_{21}) N_T$$

$$P[(\Gamma_{10} - \Gamma_{21}) N_T - \Delta N_T(\Gamma_{21} + \Gamma_{10})] = \Gamma_{10}\Gamma_{21}\Delta N_T$$

$$(P)_{\text{4level threshold}} = \frac{\Gamma_{10}\Gamma_{21}\Delta N_T}{(\Gamma_{10} - \Gamma_{21}) N_T - \Delta N_T(\Gamma_{21} + \Gamma_{10})}$$

we will assume $\Gamma_{10} >> \Gamma_{21}$ and P

$$(P)_{\text{4level threshold}} \cong \frac{\Gamma_{10}\Gamma_{21}\Delta N_T}{\Gamma_{10}N_T - \Gamma_{10}\Delta N_T} = \frac{\Gamma_{21}\Delta N_T}{N_T - \Delta N_T}$$

This results in

$$\frac{(P)_{\text{4level threshold}}}{(P)_{\text{3level threshold}}} = \frac{\Gamma_{21}\Delta N_T}{N_T - \Delta N_T}\frac{N_T - \Delta N_T}{\Gamma_{21}(\Delta N_T + N_T)} = \frac{\Delta N_T}{\Delta N_T + N_T}$$

6 Threshold of Nd :YAG, Ruby and He-Ne laser

1. Population inversion threshold

$$g(\nu) = \Delta N\sigma(\nu) = \Delta N\frac{\lambda^2 A_{12}}{8\pi}S(\nu)$$

$$g_t = -\frac{1}{2L}ln(r_1r_2) + a \approx \frac{1}{2L}(1 - r_1r_2) + a$$

$$\Delta N = \frac{g_t 8\pi}{\lambda^2 A_{12}S(\nu)}$$

$$\text{Doppler:}s(\nu_0) = \frac{1}{\delta < nu_D}\sqrt{\frac{4ln2}{\pi}}$$

He-Ne:

$$g_t = \frac{1}{2\cdot 30}\cdot(1 - 0.988\cdot 0.98) + \frac{0.002}{2\cdot 30} = 3.99\cdot 10^{-4}cm^{-1}$$

$$\Delta N_{He-Ne} = \frac{3.66\cdot 10^{-2}\cdot 8\cdot\pi\cdot 1500\cdot 10^6}{(0.6328\cdot 10^{-6})^2\cdot 1.4\cdot 10^6\cdot\sqrt{\frac{4ln2}{\pi}}}$$

$$= 2.7\cdot 10^{15}\frac{atoms}{m^3} = 2.7\cdot 10^9\frac{atoms}{cm^3}$$

Nd-Yag:

$$g_t = \frac{1}{2\cdot 10}\cdot(1 - 1\cdot 0.96) + \frac{0.03}{2\cdot 10} = 3.5\cdot 10^{-3}cm^{-1}$$

$$\sigma_{YAG}\Delta N = g_t$$

$$\Delta N_{YAG} = 3.5\cdot 10^{-3}cm^{-1}\cdot\frac{1}{3\cdot 10^{-19}}cm^{-2} = 1.16\cdot 10^{16}\frac{atoms}{cm^3}$$

Ruby:

$$g_t = \frac{1}{2\cdot 5}\cdot(1 - 1\cdot 0.96) + \frac{0.03}{2\cdot 5} = 7\cdot 10^{-3}cm^{-1}$$

$$\Delta N_{Ruby} = \frac{g_t}{\sigma_{Ruby}} = \frac{7\cdot 10^{-3}cm^{-1}}{2.7\cdot 10^{-20}cm^2} = 2.6\cdot 10^{17}\frac{atoms}{cm^3}$$

$\Longrightarrow$The population inversion for gas laser (He-Ne) is considerable lower than for solid state lasers. $\Longrightarrow$The Nd :Yag inversion population at threshold is one order of magnitude lower than for Ruby.

2. 3-level power threshold

$$\left(\frac{P}{V}\right) = \frac{1}{2} h\nu_{31} N_T \Gamma_{21}$$

Ruby:

$$\lambda = 505nm$$

$$\implies \nu_{31} = \frac{c}{\lambda} = \frac{3 \cdot 10^8}{505 \cdot 10^{-9}} = 5.94 \cdot 10^{14}$$

$$\Gamma_{21} = 500\frac{1}{s}\text{(see table)}$$

$$N_{Cr} = 2.3 \cdot 10^{19} \frac{atoms}{cm^3}$$

For a more detailed calculation of $N_{T,Cr}$ see the appendix at the end of the solution.

$$h = 6.62 \cdot 10^{-34}[J \cdot s]$$

$$\nu_{31} = 5.94 \cdot 10^{14}[\frac{1}{s}]$$

$$h\nu_{31} = 3.93 \cdot 10^{-19}[J]$$

$$\left(\frac{P}{V}\right) = \frac{1}{2} 3.93 \cdot 10^{-19} \cdot 500 \cdot 2.3 \cdot 10^{19}[\frac{W}{cm^3}] = 2260\frac{W}{cm^3}$$

$$V_{\text{Ruby,Red}} = \pi r^2 L = \pi(0.2)^2 5 = 0.63cm^3$$

$$\implies P \cong 1kW$$

3. 4-level power threshold

$$\left(\frac{P}{V}\right) = h\nu_{30} \Delta N_T \Gamma_{21}$$

$$h\nu_{30} = 1.55eV = 2.48 \cdot 10^{-19} J$$

$$\Delta N_T = 1.16 \cdot 10^{16} \frac{atoms}{cm^3}$$

$$\Gamma_{21} = 4406$$

$$\left(\frac{P}{V}\right) = 2.48 \cdot 10^{-19} \cdot 1.16 \cdot 10^{16} \cdot 4.4 \cdot 10^3 = 12.65\frac{W}{cm^3}$$

$$V = \pi(0.2)^2 \cdot 10 = 1.25cm^3$$

$$\implies P = 15.9W$$

Appendix - Density of Chromium atoms in Ruby Ruby : $Al_2O_3 \Longrightarrow$ Atomic mass: $2Al + 3O = 101.96g/mol(u)$

Chromium : $51.99g/mol(u)$

$$\frac{\text{Mass of } Cr}{\text{Mass of } Al_2O_3} = \frac{N_{Cr}[\frac{atoms}{cm^3}]51.99[\frac{u}{atom}]V[cm^3]}{N_{Al_2O_3}[\frac{atoms}{cm^3}]101.96[\frac{u}{atom}]V[cm^3]}$$

$$N_{Cr}[\frac{atoms}{cm^3}] = 0.05\% N_{Al_2O_3}\frac{atoms}{cm^3}\frac{101.96}{51.99}$$

Density of Ruby (Corundum) : $4.05g/cm^3$. Number of molecules of Ruby : 2.3910^{23} molecules.

$$\frac{4.05[g]}{101.96(\text{unified atomic mass}) \cdot 1.66 \cdot 10^{-24}[g](\text{mass of } 1/12\text{ C})} = 2.39 \cdot 10^{23} molecules$$

$$\Longrightarrow N_{Cr} = 0.05\% \frac{101.96}{51.99} 2.39 \cdot 10^{23} = 2.3\ 10^9 \frac{atoms}{cm^3}$$

7 Cavity frequency pulling effect [Reproduced from Milonni Chapter 5] We have mostly ignored the effect of refractive index of the gain medium on laser oscillation, except insofar as it enters into the equations for gain and threshold. However, it turns out that the refractive index of the gain medium actually determines to some extent the laser oscillation frequency. We will now examine how this occurs. A laser will oscillate at a frequency n such that the optical length of the cavity is an integral number of half wavelengths. That is, $L = m\lambda/2$, or $\nu = mc/2L = \nu_m$. This applies to the bare-cavity case i which the gain and refractive index of the active medium are not taken into account. In general, however, the effective optical length of a medium is not just its physical length and its refractive index $n(\nu)$. To account for the index of refraction of the active medium, therefore, we divide the cavity length into two parts $L = l + (L - l)$, where l is the length of the gain cell and remainder is empty cavity. The optical length of the gain cell is $n(\nu)l$. Thus

$$\nu = \frac{mc/2}{n(\nu)l + (L - l)} \tag{6}$$

or

$$\frac{l}{L}[n(\nu) - 1]\nu = \nu_m - \nu \tag{7}$$

Now let us assume that $n(\nu)$ is determined primarily by the single nearly resonant, lasing atomic transition. In other words, we will assume that $n(\nu)$ is essentially the resonant (or ?anomalous?) refractive index. Since other transitions contributing to the refractive index will usually be off resonance by many transition linewidths, this will often be an excellent approximation. In case of an absorbing medium the resonant refractive index is simply related to the absorption coefficient and the same applies to an amplifying (gain) medium, simply by replacing the absorption coefficient by the negative of the gain coefficient, So, for a homogeneously broadened gain medium, we have

$$n(\nu) - 1 = -\frac{\lambda_{21}}{4\pi}\frac{\nu_{21} - \nu}{\delta\nu_{21}}g(\nu) \tag{8}$$

where $\delta\nu_{21}$ is the homogeneous linewidth (HWHM). This leads to

$$\nu = \frac{\nu_{21}[cg(\nu)l/4\pi] + \nu_m\delta\nu_{21}}{[cg(\nu)l/4\pi] + \nu_{21}} \tag{9}$$

And for the cavity bandwidth ($\delta\nu_c = \frac{cg(\nu)l}{4\pi L}$) we can write

$$\delta\nu_c(\nu - \nu_{21}) = \delta\nu_{21}(\nu_m - \nu) \tag{10}$$

This shows that the actual frequency of laser is therefore pulled toward the center of the gain profile and away from the bare cavity frequency. Rewriting of the cavity frequency we can get

$$\nu = \frac{\nu_{21}\delta\nu_c/\delta\nu_{21} + \nu_m}{1 + \delta\nu_c/\delta\nu_{21}} \approx \left(\nu_{21}\frac{\delta\nu_c}{\delta\nu_{21}} + \nu_m\right)\left(1 - \frac{\delta\nu_c}{\delta\nu_{21}}\right) \approx \nu_m + (\nu_{21} - \nu_m)\frac{\delta\nu_c}{\delta\nu_{21}} \tag{11}$$

Which is the homogeneous broadening.